# Introduction to Logarithms

## Extend Your Learning
## Answers and Explanations

After watching the video, *Introduction to Logarithms,* complete the following problems.

*Introduction to Logarithms* used whole number bases for the logarithms, including base 10, which is called the common logarithm. Another logarithm, the natural logarithm, uses the number $e$ as the base. The number $e$ is a constant, and, like another famous constant $\pi$, $e$ is an irrational number. It may seem strange to use an irrational number as a base for an exponential function and for a logarithm, but $e$ is especially convenient whenever something grows or decays "continuously." For example, we might say that a city's population has doubled over the last decade. This does not mean that at the end of 10 years, people suddenly appear in the city. This growth was most likely occurring continuously. Another example is when interest on investments or loans is compounded infinitely often. In this "Extend Your Learning" we need first learn about the irrational number $e$ and then we will have a better understanding of the natural logarithm, $\log_e x$ or $\ln x$.

1. Read, and then complete the following problems to help make sense of the number $e$.

   a) We have discussed exponential growth with different bases such as 2 or 3 and have seen equations like $y = 2^t$ (for doubling) and $y = 3^t$ (for tripling). We know that when the base is smaller, the output $y$ is smaller (for the same positive $t$): $2^3 < 3^3$ We also know the larger the exponent $t$, the larger the output (for the same base): $2^3 < 2^5$ and $3^2 < 3^4$. But what happens when the base gets smaller and the exponent gets larger? Let's look at compound interest and explore!

      i) The equation we use for compound interest is $y = p(1+r)^n$ where $p$ is the amount of money originally invested, $r$ is the growth rate per time period, and $n$ is the number of time periods.

      Suppose you invest $100 and it will double (gain 100% of its value, so $r$ = 100%) after a year. If you must wait the full year to earn the interest, you will have $y = 100(1+100\%)^1$ or $y = 100(1+1)^1 = \$200$. But if the interest is paid (compounded) twice per year, then the rate for each half year is $\frac{100\%}{2}$ or $50\%$ and there are two time periods. (The base gets smaller, but the exponent grows.) The amount of money at the end of the year is $y = 100(1+\frac{1.00}{2})^2$ or $225. You have more money because the new money (the interest added after the first half year) earned interest in the second half year.

      What happens if interest is paid (compounded) three times a year?

      $100\,(1+\frac{1}{3})^3 \approx \$237$

ii) What happens if interest is paid (compounded) four times a year?

$100\,(1+\frac{1}{4})^4 \approx \$244.14$

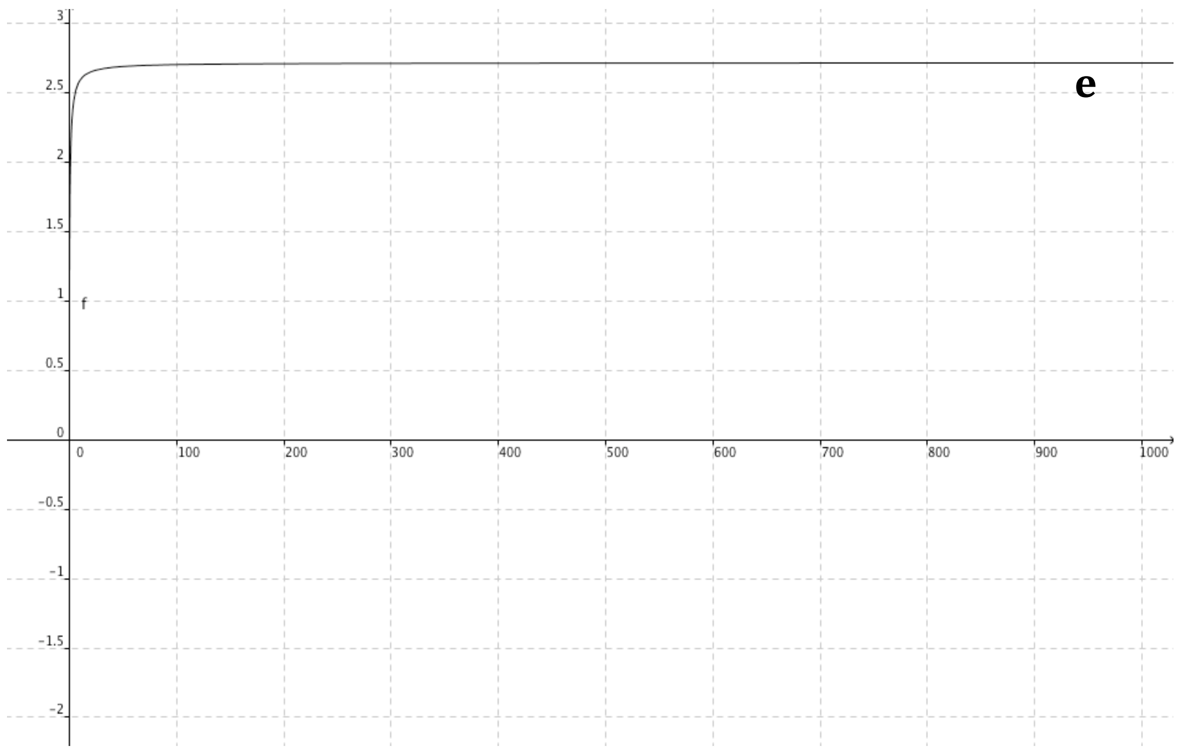
iii) Evaluate $\left(1+\frac{1}{n}\right)^n$ for successively larger values of n. What happens to the value of the expression? Record your values in the chart.

| n | $\left(1+\frac{1}{n}\right)^n$ | Value |
|---|---|---|
| 2 | $\left(1+\frac{1}{2}\right)^2$ | 2.25 |
| 3 | $\left(1+\frac{1}{3}\right)^3$ | 2.370037 |
| 4 | $\left(1+\frac{1}{4}\right)^4$ | 2.44141 |
| … | | |
| 1000 | $\left(1+\frac{1}{1000}\right)^{1000}$ | 2.71692 |
| 10000 | $\left(1+\frac{1}{10000}\right)^{10000}$ | 2.71815 |

iv) Graph $y=\left(1+\frac{1}{n}\right)^n$. Create a window to be able to see the function's behavior as $n$ increases. What is happening?

An example graph is shown below. As n gets increasingly large, the value of the function gets closer and closer to its limit, $e$.

An approximate value for $e$ is 2.718281828459045235360287.




We see from both the table and the graph that $\left(1+\frac{1}{n}\right)^n$ grows, but slower and slower. In fact, as $n$ grows without bound, $\left(1+\frac{1}{n}\right)^n$ has a limit. We call this limit value $e$. When $n$ is really, really large, $\left(1+\frac{1}{n}\right)^n$ is 2.718281828459045235360287 so this is an approximation of $e$. We found $e$ by compounding infinitely often, that is continuously.

So $e$ is the “natural base” to use in continuous growth and decay situations.

2. The natural logarithm is a logarithm with a base of $e$. It is used for problems that involve continuous growth and continuous decay. It is notated $\ln$ and just like the common logarithm $\log_{10} x = \log x$ the base does not have to be written.

$$\log_e y = x \text{ then } e^x = y$$
$$\ln y = x \text{ then } e^x = y$$

The properties of the natural logarithm are parallel to properties of the common logarithm. Complete the table with the missing entries.

| Common Logarithm | Natural Logarithm |
|---|---|
| $\log 1 = 0$ | $\ln 1 = 0$ |
| $\log(a \bullet b) = \log a + \log b$ | $\ln(a \bullet b) = \ln a + \ln b$ |
| $\log(\frac{a}{b}) = \log a - \log b$ | $\ln \frac{a}{b} = \ln a - \ln b$ |
| $n \log(a) = \log(a^n)$ | $n \ln(a) = \ln(a^n)$ |

3. Jawanza invested $100 at a bank with 2% APR (annual percentage rate) interest compounded continuously. Answer the following questions using the formula, $A=Pe^{rt}$, where A is the final amount, P is the amount invested, r is the rate, and t is the time in years.
   a. How much will his investment be worth after two years?

      $A = 100e^{0.02 \bullet 2}$

      $A = \$104.08$

   b. How long will it take for the investment to double?

      $200 = 100e^{0.02 \bullet t}$

      $\frac{200}{100} = e^{0.02 \bullet t}$

      $2 = e^{0.02 \bullet t}$

      $\ln 2 = \ln e^{0.02 \bullet t}$

      $\ln 2 = 0.02t$

      $\frac{\ln 2}{0.02} = t$

      34.66 years=t