Making Sense of Log Properties

A Closer Look at the Video

After watching the *Making Sense of Log Properties* video, make sense of the mathematics by taking a closer look at the problem situations and solutions. Use the comments and questions in bold to help you investigate the key points of the video and develop a deeper understanding of properties of logarithms.

Problem: You may use log properties, but to really understand why they work, you need to prove them. Use prior knowledge such as properties of exponents and the definition of logarithm to prove the following properties of logarithms:

$$\log(xy) = \log(x) + \log(y)$$
$$\log(x^n) = n\log(x).$$

What do we need to do if we want to prove log (xy) = log (x) + log (y)?
If we want to prove this, we need to show that the left-hand side of the equation is equivalent to the right-hand side.

Figuring out where to begin is sometimes the hardest part. We can browse through our math manual, that is, think about what we already know, to help us get started. What information do we already know?
We know the definition of logarithm, and since a logarithm is an exponent, it is also helpful to keep in mind the properties of exponents.

Mathematics Manual

Definition of a Logarithm:

If $a^x = y$ then $x = \log_a y$, and if $x = \log_a y$ then $y = a^x$.

Properties of Exponents:

$a^0 = 1$

$a^n a^m = a^{n+m}$

$\frac{a^n}{a^m} = a^{n-m}$

$(a^n)^m = a^{n.m}$

Now, where might we begin?
Since we want to prove log (xy) = log (x) + log (y), we need something with log (x) and log (y). We can try letting n = log x and m = log y. Then, using the definition of logarithms, if n = log x then $x = 10^n$; this is also true for $y = 10^m$.

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$\underline{\log (xy) = \log (x) + \log (y)}$

Let $n = \log (x)$ and $m = \log (y)$

$x = 10^n$ and $y = 10^m$

Since we're working with the log (xy), what would be a reasonable next step in our proof?
We should multiply x and y. $xy = (10^n)(10^m)$. Then, using the multiplication property of exponents, $xy = 10^{(n + m)}$.

$\underline{\log (xy) = \log (x) + \log (y)}$

Let $n = \log x$ and $m = \log y$

$x = 10^n$ and $y = 10^m$

$xy = (10^n)(10^m)$

$xy = 10^{(n + m)}$

Now we have x times y, but we want an equation involving logarithms. What can we do to the equation, $xy = 10^{(n + m)}$ to change it into an equation with a logarithm?
We should use the definition of logs again to write $\log (xy) = n + m$.

$\underline{\log (xy) = \log (x) + \log (y)}$

Let $n = \log x$ and $m = \log y$

$x = 10^n$ and $y = 10^m$

$xy = (10^n)(10^m)$

$xy = 10^{(n + m)}$

$\log (xy) = n + m$

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Think back to how we started the proof. How did we define n and m?
We let n = log x, and m = log y.

How can we use that information to finish the proof?
We can substitute those expressions and get log (xy) = log (x) + log (y). Now, we have proven our property.

$\underline{\log (xy) = \log (x) + \log (y)}$

Let $n = \log x$ and $m = \log y$

$x = 10^n$ and $y = 10^m$

$xy = (10^n)(10^m)$

$xy = 10^{(n + m)}$

$\log (xy) = n + m$

$\log (xy) = \log x + \log y$

What connection is there between multiplying numbers with exponents and the logarithm of a product?
When you multiply numbers with exponents, you add the exponents. Similarly, when you take the log of a product, you add the logs.

Property of Exponents:

$a^n \cdot a^m = a^{n+m}$

Property of Logarithms:

$\log (xy) = \log (x) + \log (y)$

Our math manual is continually growing by adding what we learn to what we already know.
Let's see if using our updated math manual can help us prove one more property: $\log (x^n) = n \log (x)$.

What is another way $\log (x^n)$ can be written that relates to the property of logarithms we have already proved?

If you see log (x^n), you might quickly recognize it is the same as the log of the quantity $x \cdot x \cdot \ldots \cdot x$, taken n times. Using the previous property, we know that the log of $x \cdot x \cdot \ldots \cdot x$, n times, is equal to log x + log x + … + log x, n times or n log (x).

Unfortunately, we cannot use this as a proof because this only makes sense for whole number values of n. **In proofs, we have to choose strategies that can work for any value.** Let's not get discouraged though; we'll just consult our manual and try again.

If we want to start the proof of this property in a similar way to the first proof, what should we do first?
Let m = log x, and use the definition of logarithm, to write $10^m = x$.

<u>log (x^n) = n log (x)</u>

Let m = log x

$10^m = x$ → definition of logarithm

Since we are trying to prove something about a power, what might be a logical next step in our proof?
We can raise both sides of the equation to the nth power. Then, we can use a property of exponents for powers to get $10^{m \cdot n} = x^n$.

<u>log (x^n) = n log (x)</u>

Let m = log x

$10^m = x$

$(10^m)^n = x^n$ → raise both sides to nth power

$10^{m \cdot n} = x^n$ → property of exponents

The property we are trying to prove has log (x^n). How can we get from our current equation, $10^{(m \cdot n)} = x^n$ to an equation with log (x^n)?
Using the definition of logarithms again, we can write $m \cdot n = \log(x^n)$.

$\underline{\log (x^n) = n \log (x)}$

Let $m = \log x$

$10^m = x$

$(10^m)^n = x^n$

$10^{m \cdot n} = x^n$

$\mathbf{m \cdot n = \log (x^n)}$ → definition of logarithms

At the beginning of the proof we let m = log x. How can we use this to finish the proof? By substituting, we have $\log (x) \bullet n = \log x^n$ or $n \bullet \log (x) = \log x^n$. Now, we can add this property to our math manual.

$\underline{\log (x^n) = n \log (x)}$

Let $m = \log x$

$10^m = x$

$(10^m)^n = x^n$

$10^{m \cdot n} = x^n$

$m \cdot n = \log (x^n)$

$\mathbf{n \log x = \log (x^n)}$

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